THE INFIMA AND SUPREMA OF FAMILIES OF TOPOLOGIES*

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1. Introduction.

Let R and I be two nonempty sets and let I consist of two or more elements. For each $\mu\in I$, let τ_μ be a topology for R. Then the inf of the set of topologies, $\wedge\{\tau_\mu:\mu\in I\}$ is a topology for R, called the inf topology. The sup of these topologies, $\vee\{\tau_\mu:\mu\in I\}$ is also a topology for R, the sup topology.

Norman Levine indicated in (2) that the family $\{\mathcal{J}_\alpha:\alpha\in I\}$ of topologies for X generates a topology $\mathcal{J}$ for X in the following natural way: a subset O of X is in $\mathcal{J}$ iff for each $x\in O$, there exist $\alpha_1,\cdots\cdots,\alpha_n$ in I and $U_i\in\mathcal{J}_{\alpha_i}$ such that $x\in U_1\cap\cdots\cdots\cap U_n\subset O$. (See (2)). However, this topology is none other than the inf topology for R if we take $R=X$.

In this paper the author provides some properties of inf topologies and related group topologies in §2, and then provides some properties among inf and sup topologies in §3.

2. Inf topologies and related group topologies.

In the following theorems we designate by τ the inf topology $\wedge\{\tau_\mu:\mu\in I\}$.

Let (D^*, τ^*) be the diagonal of the product space $(\times R, \times\tau_\mu)$, $\mu\in I$, where τ^* is the topology induced by the product topology $\times\tau_\mu$ on the diagonal D^*. Here $\xi^*\in D^*$ iff $\xi^*:I\to R$ is a constant.

THEOREM 1. (R, τ) is homeomorphic to (D^*, τ^*).
Note. This theorem is the fundamental theorem provided by Levine in (2).

THEOREM 2. For each $\mu\in I$, (R, τ_μ) is a continuous, one-to-one image of (R, τ).

Proof. Let φ_μ: $(R, \tau)\to(R, \tau_\mu)$, $\mu\in I$, be a map defined by $\varphi_\mu p=p$ for all $p\in R$. Then φ_μ is evidently one-to-one and onto. If $\tau_1\leq\tau_2$, then every τ_2-open set is τ_1-open. Hence φ_μ is continuous by $\tau\leq\tau_\mu$ for every $\mu\in I$.

THEOREM 3. If D^* is closed, then (R, τ) is T_2.
Note. This theorem is provided by Levine in (2).

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A topological space (R, τ) will be called a C-C space iff the closed sets in R coincide with the compact sets in R.

THEOREM 4. If D^* is compact and if every (R, τ_μ) is C-C, then (R,τ) is C-C
Proof. The hypotheses imply that D^* is C-C, so (R,τ) is C-C by Theorem 1.

Let (R^+, τ^+) be the one-point compactification of (R, τ) and let $(R^+, \tau_\mu{}^+)$ be the one-point compactification of (R, τ_μ) for each $\mu \in I$. Then we have the following corollary.

COROLLARY 5. If D^* is closed and if every (R, τ_μ) is compact T_2, then (R^+, τ^+) is homeomorphic to $(R^+, \tau_\mu{}^+)$ for each $\mu \in I$.

Proof. The hypotheses imply that (R, τ) is compact T_2, so (R, τ) is, by Theorem 2, automatically homeomorphic to (R, τ_μ) for each $\mu \in I$. Therefore the one-point compactifications are homeomorphic.

COROLLARY 6. If the product space $(\times R, \times\tau_\mu)$ is C-C and if every (R, τ_μ) satisfies the first axiom of countability, then (R, τ) is homeomorphic to (R, τ_μ) for each $\mu \in I$, provided D^* is closed.

Proof. If the product space $(\times R, \times\tau_\mu)$ is C-C, so is (R, τ_μ) for each $\mu \in I$, and since every (R, τ_μ) satisfies the first axiom of countability, (R, τ_μ) is compact T_2 for each $\mu \in I$ by Levine's theorems in (3).

THEOREM 7. If (G, τ_μ) is a topological group for each $\mu \in I$, then (G, τ) is also a topological group.

Proof. It is clear that (G, τ) is a subgroup of $(\times G, \times\tau_\mu)$, a topological group, and hence is a topological group under the induced topology.

We will denote by (G, τ_μ) a topological group for each $\mu \in I$ in the following theorems.

THEOREM 8. If every (G, τ_μ) is compact and satisfies the second axiom of countability, then (G, τ) is isomorphic to (G, τ_μ) for each $\mu \in I$, provided D^* is closed and I is countable.

Proof. By Theorem 2, (G, τ) is homomorphic to each (G, τ_μ). Under the hypotheses, (G, τ) and all (G, τ_μ) are compact topological groups satisfying the second axiom of countability, so the homomorphic map is open. Hence (G, τ) is isomorphic to (G, τ_μ) for each $\mu \in I$.

COROLLARY 9. If the product space $(\times G, \times\tau_\mu)$ is C-C and if D^* is closed, then (G, τ) is isomorphic to (G,τ_μ) for each $\mu \in I$.

Proof. The hypotheses imply that each (G, τ_μ) is compact T_1 by Levine's theorems in (3), then (G, τ_μ) is compact T_2 for each $\mu \in I$.

3. Inf and sup topologies.

In this section we designate by τ the inf topology $\wedge\{\tau_\mu:\mu\in I\}$ and designate by τ' the sup topology $\vee\{\tau_\mu:\mu\in I\}$.

THEOREM 10. (R, τ') is a continuous, one-to-one image of each (R, τ_μ) and (R, τ).

Proof. By $\tau\leq\tau_\mu\leq\tau'$, the theorem holds evidently.

THEOREM 11. If at least one of the τ_μ is T_i, $(i=0,1,2)$, then (R, τ) is also T_i. If τ' is T_i, $(i=0,1,2)$, then (R, τ) as well as every (R, τ_μ) is T_i.

Proof. If $\tau_1\leq\tau_2$ and τ_2 is a T_i-topology, then so is τ_1. Since $\tau\leq\tau_\mu$ for every $\mu\in I$, (R, τ) is T_i if at least one of the τ_μ is T_i. If τ' is T_i, then since $\tau\leq\tau_\mu\leq\tau'$, (R, τ) and all (R, τ_μ) are also T_i.

THEOREM 12. If at least one of the τ_μ is compact (connected), then (R, τ') is also compact (connected). If τ is compact (connected), then (R, τ') as well as every (R, τ_μ) is compact (connected).

Proof. If $\tau_1\leq\tau_2$, then every τ_1-compact (τ_1-connected) set is also τ_2-compact (τ_2-connected). The theorem holds, since $\tau\leq\tau_\mu\leq\tau'$.

THEOREM 13. If (R, τ) is compact and if (R, τ') is T_2, then (R,τ), (R, τ') and all (R, τ_μ) are homeomorphic to each other.

Proof. The hypotheses imply that (R, τ), (R, τ') and all (R, τ_μ) are compact T_2 by Theorem 11 and Theorem 12. Hence they are homeomorphic to each other.

A set M of a topological space R will be termed strongly connected (written henceforth as s.c.) iff M⊆U or M⊆V whenever $M\subseteq U\cup V$, U and V being open sets in R.

THEOREM 14. If each (R, τ_μ) is s.c. and if D* is closed, then both (R,τ) and (R, τ') are s.c.

Proof. By Levine's theorems in (4), the hypotheses imlply that $(\times R, \times\tau_\mu)$ is s.c., and D* is also s.c., so (R, τ) is s.c. by Theorem 1. Also, (R, τ') is s.c. by Theorem 10.

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ABSTRACT: Let R and I be two nonempty sets and let I consist of two or more elements. For each $\mu\in I$, let τ_μ be a topology for R. Then the inf of the set of topologies, $\wedge\{\tau_\mu:\mu\in I\}$ is a topology for R, and the sup of these topologies, $\vee\{\tau_\mu:\mu\in I\}$ is also a topology for R. The author provides some properties of inf topologies and related group topologies, and also the properties among inf topologies and sup topologies in this paper.

拓樸族之最大下界及最小上界

吳 青 木

摘要:命R與I爲二個非空虛集合，且I含有二個或二個以上之元素。對於每一個I中之元素μ，命τ_μ爲R之一個拓樸，則R之拓樸族所構成之最大下界及最小上界皆成爲R之拓樸。筆者於此篇論文中，給與拓樸族之最大下界與其有關之群拓樸，以及最大下界與最小上界之間之一些性質。